Commitment, Welfare, and the Frequency of Choice*

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Abstract

This paper studies the *frequency of choice*—the number of times a choice is made in a dynamic setting—when individuals have dynamically inconsistent preferences. In some prominent dynamic examples (like the choice between investing in a project now or later, doing an unpleasant task now or procrastinating it until later, and eating a cake), we find that the commitment allocation can multiself Pareto dominate the non-cooperative equilibrium allocation *only if* the number of time-dated selves (the frequency of choice) is high enough. We conclude that the frequency of choice is an important consideration in behavioral welfare analysis.

Key words: Frequency of Choice, Commitment, Behavioral Welfare Analysis, Dynamic Inconsistency, Hyperbolic Discounting.

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1 Introduction

In dynamic choice settings, people often change their minds and fall short of their initial goals (Strotz (1956)). For example, an individual may procrastinate an unpleasant task or may save less for the future than he had initially hoped. In order to study this type of behavior, it has become common in economic research to assume individuals have dynamically inconsistent preferences.

The literature on dynamically inconsistent preferences makes a wide range of assumptions about the *frequency of choice*, defined here to be the number of times a given choice is made in a dynamic setting. In some models individuals make just two or three sequential choices, but in others they make many choices, or even infinitely many choices. Researchers typically make assumptions about the frequency of choice out of convenience, rather than for some fundamental economic reason. For instance, a three-period model may be used to illustrate a point or to derive an analytical result. Likewise, an infinite-horizon model may be convenient for the same reason. And in quantitative studies where the goal is to make the most precise calculations possible, the researcher may choose the highest frequency allowed by computational constraints. While all of these motivations for a given frequency are quite sensible, less is specifically known about the economic implications of the frequency of choice in dynamically inconsistent models.

Our objective is to examine whether the frequency of choice is an important consideration in behavioral welfare analysis. We find that it can be. In fact, in some prominent dynamic settings that are indicative of common trade-offs facing individuals, the frequency of choice itself determines whether or not standard behavioral welfare analysis is consistent with a multiself Pareto criterion.

In a series of simple examples, we compare the commitment allocation (the sequence of decisions that the time-zero self would like to impose on future selves) to the equilibrium allocation (the sequence of decisions that emerge as the non-cooperative equilibrium of the intra-personal game). We assume individuals discount the future hyperbolically and are sophisticated, and we say that the commitment allocation multiself Pareto dominates the equilibrium allocation if all of the time-dated selves of a single individual prefer the commitment allocation over the equilibrium allocation.[1] Our main finding is that the commitment allocation can multiself Pareto dominate the equilibrium allocation *only if* the number of time-dated selves (the frequency of choice) exceeds a threshold.[2]

We study two types of finite-horizon settings. First, we consider a *stationary* setting in which a given

[1] Under the alternative assumption of naiveté, a given self anticipates a different future consumption allocation than what he actually experiences. Thus, it is a philosophical question of how to even define the welfare of a given time-dated self in this case: should it be based on anticipated or experienced future consumption? We do not attempt to address this question here.

[2] This does not imply that all selves think that commitment is optimal. Instead, while some selves find commitment to be suboptimal, they all agree that it beats the equilibrium allocation.

choice repeats itself over and over until the end of the finite time horizon. One example of a stationary setting is a fruit tree that bears new fruit each period. An individual faces a repeated choice between eating fruit now when it tastes good or leaving the fruit on the tree for one extra period to fully ripen. An equivalent example is Laibson's (2003) investment problem in which the individual is endowed each period with income that can be consumed immediately or invested in a one-period project that pays a positive return, or an individual who must repeatedly choose between doing an unpleasant job now or procrastinating it until later, as in O'Donoghue and Rabin (1999). Second, we consider a *non-stationary* setting in which the choice set changes over time, as in a non-renewable resource problem like eating a cake (Hall (1998)). In both the stationary and non-stationary settings we quantify the frequency of choice that is required for the commitment allocation to multiself Pareto dominate the equilibrium allocation, and we find that the threshold can be quite small (in the single digits), depending on the parameterization of the particular model.

The intuition for these results is as follows. If there is a small number of selves, then a given self may have the power to significantly influence the equilibrium allocation. In this case, the equilibrium allocation may be relatively close to what that self wants. However, if there is a large number of selves, then the power to influence the equilibrium allocation is diffuse, making the equilibrium relatively dissimilar to the desired allocation of a given self. Therefore, it becomes possible to find allocations that Pareto dominate the equilibrium. In other words, although a large number of selves means that there are many conflicting points of view on how resources should ideally be allocated, it also means that the equilibrium is far away from the wishes of all selves, and this creates space for a Pareto improvement.

A caveat is in order. Because a choice is made in each period in typical discrete time models like those we have described above, the frequency of choice and the total number of periods are the same. Therefore, it is unclear whether it is really the frequency of choice or just the total length of model time that drives our welfare conclusions. To identify the source of our results, we extend our non-stationary (cake eating) setting to a continuous time setting in which we distinguish between the frequency of choice and the total length of time. In this extension, time is continuous and the total length of time is fixed, while the frequency of choice is intentionally discrete (finite) so that we can vary the total number of choices made while holding fixed the overall length of time. Essentially, we are checking to see if simply partitioning time into smaller decision increments will generate the same welfare conclusions that we are drawing from discrete time models, and we conclude that the answer is yes.

We do not claim that all of our results are fully general. Like any model, our settings are stylized in a variety of ways and the normative conclusions that we draw from them are potentially limited.

Nevertheless, the settings that we consider are well known, and our results suggest that the frequency of choice can play a fundamental role in welfare analysis.

Our paper is connected to a large literature on behavioral welfare analysis, in which the standard approach is to assume that the welfare of an individual with dynamically inconsistent preferences is maximized if he can *commit* to his initial goal.[3] This approach is based on the principle that people judge correctly their own best interests when they plan for the future, and a failure to exercise self control is an error in decision making. Under this philosophy, the goal of a policymaker is to maximize the utility of the time-zero self, and outcomes are evaluated based on how closely they align with an individual's initial goals. However, despite the intuitive appeal of the time-zero welfare criterion, some researchers have expressed the concern that committing an individual to his initial goal unfairly hurts later selves.[4] Our analysis suggests that this concern may be more relevant for *low-frequency* models than for *high-frequency* models, since in the latter case all of the time-dated selves of an individual may prefer commitment over the equilibrium.

Finally, some studies document that commitment may be Pareto improving. For example, Laibson (1996) shows that the saving rate that the time-zero individual would commit himself to follow in an *infinite-horizon* setting is Pareto superior to the equilibrium saving rate (see also Goldman (1979)).[5] While Hall (1998) showed that Laibson's result does not hold in a three-period model, less is known specifically about how the frequency of choice may affect such conclusions. In this paper we quantify the threshold frequency that is necessary to deliver the result that commitment Pareto dominates the equilibrium, and in particular we find that this threshold can be quite low.

2 Notation and definitions

Time is discrete and indexed by a finite number of selves or nodes $t = 0, 1, 2, ..., T$. Nothing happens at $t = 0$; no decisions are made and no economic activities occur (e.g., no consumption, no income received,

[3] For example, see Laibson (1996, 1997, 1998), Laibson, Repetto, and Tobacman (1998), O'Donoghue and Rabin (1999, 2000, 2001, 2003, 2007), Gruber and Kőszegi (2001, 2004), DellaVigna and Malmendier (2004), Heidhues and Kőszegi (2009, 2010), among many others. See Brocas, Carrillo, and Dewatripont (2004) and Bryan, Karlan, and Nelson (2010) for surveys.

[4] For instance, Gul and Pesendorfer (2004, p.263) point out that welfare analysis based on time-zero preferences "has the planner forever guarding the perceived interests of the nonexistent former selves," which they later describe as "odd" (Gul and Pesendorfer (2008, p.38)). Likewise, Rubinstein (2006, p.248) states "One criticism made of behavioral economics is the arbitrariness of the welfare criterion...why should the utility of the first self be the basis for welfare considerations?" And Brocas, Carrillo, and Dewatripont (2004, p.51) state that "there is no normative foundation" for equating welfare with time-zero preferences. See Bernheim and Rangel (2009) for additional discussion.

[5] In addition, there is a literature that computes the welfare gains from committing to specific rules and participating in specific policies. For instance, see Laibson (1997), Laibson, Repetto, and Tobacman (1998), Cropper and Laibson (1999), and İmrohoroğlu, İmrohoroğlu, and Joines (2003).

etc.). It is an inaction node. It is there simply to allow us to consider what self 0 would like his future selves to do. All of the other nodes $t > 0$ are action nodes or decision nodes, and we refer to T as the total number of decision nodes or the **frequency of choice**.

An **allocation** is a vector of consumption decisions $\mathbf{c} = \{c_1, c_2, ..., c_T\}$. The set of feasible allocations is $\mathbf{C}$. Following Caplin and Leahy (2004), lifetime utility is a mapping $U(t, \mathbf{c}) : \mathbb{R}^T \mapsto \mathbb{R}$ that depends on the vantage point $t \in [0, T]$. Note that this definition is general enough to include any assumption about how the individual values past consumption, including the common special case in which he places no value on past consumption, which is the case that we will focus most (though not all) of our attention on. Following the terminology of Bernheim and Rangel (2009), an allocation $\mathbf{c}' \in \mathbf{C}$ **strictly multiself Pareto dominates** another allocation $\mathbf{c}'' \in \mathbf{C}$ if and only if $U(t, \mathbf{c}') > U(t, \mathbf{c}'')$ for all $t \in [0, T]$.

The **commitment allocation** $\mathbf{c^0}$ is the optimal allocation from the vantage point of self 0, $\mathbf{c^0} = \arg\max_{\mathbf{c}\in\mathbf{C}} U(0, \mathbf{c})$. The **equilibrium allocation** $\mathbf{c}^*$ is the allocation that actually materializes from the internal conflict among the many time-dated selves who each have a different view on optimal decision making. We assume individuals are **sophisticated** and are therefore fully aware of the choices that future selves will make.

Dynamic inconsistency is the situation in which $\mathbf{c^0} \neq \mathbf{c}^*$. The thrust of this paper is to understand if the frequency of choice affects whether $\mathbf{c^0}$ strictly multiself Pareto dominates $\mathbf{c}^*$. Of course, even though $\mathbf{c^0}$ is Pareto optimal and $\mathbf{c}^*$ typically is not, there is no guarantee that $\mathbf{c^0}$ strictly multiself Pareto dominates $\mathbf{c}^*$ because this requires that all selves be made better off in moving from $\mathbf{c}^*$ to $\mathbf{c^0}$. We show below that the possibility for such a result may hinge on the frequency of choice.

3 Stationary examples

We develop a variety of analytical and quantitative results in this section. To be concrete in language, we will consider a simple example—eating fruit from a tree—although our setting is generic enough to apply to a variety of repeated trade-offs that decision makers face.

An individual plants a tree at $t = 0$. At each subsequent node (beginning at $t = 1$) the tree bears exactly one piece of new fruit. The fruit may be consumed immediately or it may be left on the tree an additional period to fully ripen. Either way, the fruit tastes good, but it tastes better if it is left to fully ripen. The fruit is totally rotten if it is left on the tree for more than one additional period. The last piece of new fruit is produced at $t = T - 1$, which may be consumed immediately or consumed one period later at $t = T$, after which the tree dies and no consumption takes place beyond T.

At each age $t \in [1, T-1]$ the individual faces a simple choice. Take a small amount of utility now c^- or a larger amount c^+ one period later. The choice repeats itself over and over, for all $t \in [1, T-1]$. We are assuming utility is linear at this point, which allows us to collapse the dynamic programming problem into a sequence of independent, static problems. In a subsection later (and also in our quantitative model), we study concave utility.

3.1 β discounting

Following Laibson (2003) and many others, a given self t applies the following once-off discount sequence to future utility $\{1, \beta, \beta, \beta, ...\}$ with $\beta < 1$.[6] For now, we assume the individual does not value past consumption (we will relax this assumption later in the paper).

From the perspective of $t = 0$, the individual wants to always be patient and eat ripe fruit, c^+ at $t \in [2, T]$, because there is no perceived cost of waiting for the fruit to ripen. Hence the ideal allocation of self 0 is

$$c_1 = 0, \text{ and } c_t = c^+ \text{ for } t \in [2, T],$$

or written compactly, $\mathbf{c^0} = (0, c^+, ..., c^+)$.

Likewise, from the perspective of any $t > 0$, the individual would want future selves to be patient at all nodes $t + 1$ and beyond, but he will be impatient at the current time if

$$c^- > \beta c^+ \Longrightarrow \beta < \frac{c^-}{c^+}.$$

We assume this condition is met so that we can have a meaningful discussion about dynamic inconsistency. Hence, self t will be impatient at the current moment and eat unripened fruit. And there is no way for him to impose on later selves his preference for patience at later dates, nor do past choices affect the options currently available to him. Hence, he recognizes that, in equilibrium, all selves will choose to eat the fruit immediately rather than let it ripen. The equilibrium allocation therefore is

$$c_t = c^- \text{ for } t \in [1, T-1], \text{ and } c_T = 0,$$

or written compactly, $\mathbf{c}^* = (c^-, ..., c^-, 0)$.

We are now ready to state some analytical results. All proofs can be found in Appendix A.

[6] This is a simple way to capture "present-biased" dynamically-inconsistent preferences. This is a typical calibration (Laibson (2003)) of quasi-hyperbolic discounting, and its simplicity allows us to derive a variety of analytical results. We will pursue the more generic $\beta\delta$ form in the next subsection of the paper.

Proposition 1 *Commitment can't multiself Pareto dominate the equilibrium for the special case of $T = 2$. However, if $T > 2$ then there exists a non-empty, open convex subset S of $\mathbb{R}^+$ such that commitment will strictly multiself Pareto dominate the equilibrium for all $\beta \in S$.*

Proposition 1 allows one to make a Pareto argument for the time-zero welfare criterion, at least within the context of this specific example. While it is true that the commitment allocation will never multiself Pareto dominate the equilibrium allocation when there are only two decision nodes, that fragile result is overturned if we simply increase the number of decision nodes to anything higher than two. As long as $T > 2$ we can always find a β for which the commitment allocation will multiself Pareto dominate the equilibrium allocation, and in such a case it would seem reasonable to advocate for commitment.

Proposition 2 *The size of the set S is monotonically increasing in T and the lower bound of S converges to zero as T goes to infinity.*

Proposition 2 further strengthens the argument for the time-zero approach, because it states that the larger is T, the larger is the parameter space S that can deliver a Pareto gain from commitment. In fact, the commitment allocation multiself Pareto dominates the equilibrium allocation for *any* $\beta < c^-/c^+$ if the tree lives forever. Figure 1 is a graph of the non-empty set S that delivers the Pareto result. Note the convexity of the lower bound—the parameter space opens up rapidly as T increases. This is true for any assumptions about the other parameters (c^- and c^+).

The following is an immediate implication of Propositions 1 and 2.

Corollary 3 *The commitment allocation will multiself Pareto dominate the equilibrium allocation if and only if the total number of decision nodes in the choice problem exceeds a threshold.*

Corollary 3 states that for any parameterization of β that satisfies the dynamic inconsistency restriction ($\beta < c^-/c^+$), there is a threshold in the total number of decision nodes, above which the commitment allocation will multiself Pareto dominate the equilibrium allocation and below which self 1 prefers the equilibrium. So for a given β, one can always simply increase the total number of decision nodes in this choice problem to ensure that all selves prefer commitment over the equilibrium.

While this may seem like a paradoxical result, it has an intuitive explanation. When there is a small number of decision nodes (i.e., a small number of selves), a given self may have the power to significantly influence the equilibrium allocation. If so, then the equilibrium allocation may not be so bad from his perspective; in fact, it may be relatively close to what he wants. However, when there is a large number of selves, then the power to influence the equilibrium allocation is diffuse among selves. No one self has much power to influence the equilibrium outcome and hence the equilibrium is relatively dissimilar to the desires of a given self. And in such an environment where power is diffuse and no self gets what he wants, it is possible to find allocations that Pareto dominate the equilibrium allocation. In other words, just because the many selves disagree on the ideal allocation of resources doesn't mean that they can't all agree that certain allocations are better than others; and, the further the equilibrium allocation gets from any one self's ideal allocation, the more room there is for such an agreement.

Let's take a deeper look at why the frequency of choice matters. Self 1 is the "holdup self" which means that if he prefers commitment over the equilibrium, then all of the selves will prefer commitment over the equilibrium. Consider how his ideal allocation $\mathbf{c^1}$ compares to $\mathbf{c^0}$ and $\mathbf{c}^*$. For $T > 2$,

$$\mathbf{c^0} = (0, c^+, ..., c^+), \quad \mathbf{c^1} = (c^-, 0, c^+, ..., c^+), \quad \mathbf{c}^* = (c^-, ..., c^-, 0).$$

Self 1 disagrees with self 0's ideal allocation only at nodes $t = 1$ and $t = 2$. On the other hand, self 1 disagrees with the equilibrium allocation at every node except $t = 1$. In other words, as T increases, the equilibrium allocation starts to look less and less like the desires of self 1 and the commitment allocation starts to look more and more like the desires of self 1. This pattern can be seen in Figures 2 and 3, which show the specific cases of $T = 2$ and $T = 6$ just to illustrate this point.

To get a feel for the magnitude of the threshold number of decision nodes needed to produce the Pareto result, suppose $c^- = 1/2$ and $c^+ = 1$ (the fruit tastes twice as good if left to ripen). If the individual discounts the future extremely heavily (say $\beta = 0.1$) then there must be 11 or more nodes in order for the holdup self (self 1) to prefer to always be patient and eat ripe fruit. But at more modest levels of discounting (say $\beta = 0.4$), then just 3 nodes is enough to ensure that the holdup self prefers to always be patient and wait for the fruit to ripen. That is, in this particular parameterization, just 3 nodes is enough to ensure that all selves would prefer the commitment allocation of always waiting to let the fruit ripen over the equilibrium allocation of always eating unripe fruit. Alternatively, suppose $c^- = 3/4$ and $c^+ = 1$. Then there must be 29 or more decision nodes when $\beta = 0.1$ and there must be 6 or more decision nodes when $\beta = 0.4$.

Our focus thus far on the special case in which the individual does not derive any utility from past consumption is an innocuous simplification. For instance, suppose the individual discounts past utility using the discount sequence $\{1, \gamma, \gamma, \gamma, ...\}$, all of the theoretical conclusions above continue to hold for any $\gamma \in [0, 1)$.

Proposition 4 *Propositions 1 and 2 and Corollary 3 continue to hold if, in addition to discounting future utility with discount factor β, the individual applies discount factor $\gamma \in [0, 1)$ to past consumption.*

As a second robustness check, we note that sometimes in the literature and in the real world, time starts at $t = 1$ instead of $t = 0$. That is, the individual must make a consumption decision at the beginning of time, without the time-zero inaction node. In this case, the commitment allocation is self 1's ideal allocation $\mathbf{c^1}$. This does not disrupt our results at all, except to push everything back a period.

Proposition 5 *If time starts at $t = 1$ (instead of $t = 0$) and a consumption choice is made at $t = 1$, then as long as $T > 3$ there exists a non-empty, open convex subset S of $\mathbb{R}^+$ such that commitment will strictly multiself Pareto dominate the equilibrium for all $\beta \in S$.*

A final point to consider is how the time-zero commitment allocation compares to other commitment allocations. That is, suppose a commitment technology is made available to self $t \in [1, T-2]$, and suppose the individual follows the equilibrium allocation before this commitment technology is made available. The individual is sophisticated and therefore early selves know the future date t at which commitment will occur. (Note that commitment is already naturally available to self $T-1$ because the final self is left with whatever self $T-1$ decides to leave him). For $t \in [1, T-2]$, the allocation that self t would choose for his current and future selves is c^- at t, then 0 at $t+1$, and c^+ on $[t+2, T]$. We refer to this allocation as the self-t commitment allocation.

So the question is whether it is possible that all selves prefer the self-0 commitment allocation over the self-t commitment allocation. We state the answer in the following proposition.

Proposition 6 *There is a non-empty set of β values for which the self-0 commitment allocation multiself Pareto dominates all other self-t commitment allocations for $t > 1$. The self-0 commitment allocation does not, however, multiself Pareto dominate the self-1 commitment allocation.*

Thus, if a policymaker can provide a commitment device at any point, there would be no reason to delay commitment beyond self 1. The only ambiguity would be whether to give the device to self 0 or wait a period and give it to self 1.

3.2 Other examples: investing in a project; procrastinating an unpleasant task

Notice that while we will speak of eating fruit from a tree, this setting is generic enough to illustrate the types of trade-offs that are common in other problems. The investment problem described in Laibson

(2003) is an example. At each $t = 1, ..., T-1$, the individual is endowed with income y that can be consumed immediately or invested in a one-period project with gross return R. This choice repeats itself again and again until the last decision is made at $T-1$. Setting $y = c^-$ and $R = c^+/c^-$ replicates our fruit tree setting.

Moreover, all of the above results readily extend to an alternative setting in which an individual must do an unpleasant job today or next period. Now, dynamically inconsistent preferences cause procrastination (rather than immediate action as in the fruit-tree example). If done today, the job causes disutility j, but if he procrastinates until next period the job grows more difficult and causes disutility Rj, with $R > 0$. Under β discounting, the individual will procrastinate if $j > \beta Rj$, or $\beta < 1/R$, which we assume. Suppose that the job repeats itself over and over, with the first job appearing at $t = 1$ (which can be completed at $t = 1$ or $t = 2$) and the last job appearing at $t = T-1$ (which can be completed at $t = T-1$ or $t = T$). The commitment allocation is $(j, ..., j, 0)$ and the equilibrium allocation is $(0, Rj, ..., Rj)$. Selves 2 and beyond prefer commitment, and self 1 prefers commitment as long as $\beta \in ([(T-1)(R-1)+1]^{-1}, 1/R)$ which is non-empty for all $T > 2$, and the lower bound of the set is decreasing in T and limits to zero as $T \to \infty$. Hence for a given $T > 2$ there is always a range of β for which the commitment allocation multiself Pareto dominates the equilibrium allocation; and inversely, for a given β we can always reach the conclusion that commitment multiself Pareto dominates the equilibrium if we increase the total number of decision nodes.

3.3 $\beta\delta$ discounting

We now consider the popular quasi-hyperbolic discount function $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$. From the perspective of $t = 0$, unripe fruit at t confers utility $\beta\delta^t c^-$ and a ripe piece of fruit at $t+1$ confers utility $\beta\delta^{t+1}c^+$. The time-zero self would like to commit his time t self to be patient if $\beta\delta^{t+1}c^+ > \beta\delta^t c^-$, or $\delta > c^-/c^+$. We assume this condition is satisfied. Likewise, when time t arrives, the individual will act in a dynamically inconsistent fashion if $c^- > \beta\delta c^+$ or $\beta < c^-/(\delta c^+)$ and we likewise assume this condition is satisfied so that we can have a meaningful discussion about dynamic inconsistency. Under these conditions, the commitment and equilibrium allocations are the same as in the previous case where $\delta = 1$, hence

$$\mathbf{c^0} = (0, c^+, ..., c^+), \quad \mathbf{c}^* = (c^-, ..., c^-, 0),$$

and it is still the case that self 1 is the holdup in drawing the conclusion that commitment multiself Pareto dominates the equilibrium.

Relative to β discounting, we obtain almost identical welfare results. As in Proposition 1, the commitment allocation can't strictly multiself Pareto dominate the equilibrium allocation for the special case of $T = 2$, but as long as $T > 2$ then there exists a non-empty, open convex subset S of $\mathbb{R}^+$ such that commitment will multiself Pareto dominate the equilibrium for all $\beta \in S$.

As in Proposition 2, the size of the set S is monotonically increasing in T because the lower bound is decreasing in T. However, unlike the case of β discounting in which the lower bound of S limits to zero, under $\beta\delta$ discounting the lower bound limits to a value above zero.

This does not interfere with the conclusion that for a given $T > 2$ there is always a range of β that will deliver the Pareto result. That result holds because S is non-empty for $T > 2$. However, it does place restrictions on the inverse of that result (Corollary 3): for a given β, commitment will multiself Pareto dominate the equilibrium if and only if the total number of decision nodes in the choice problem exceeds a threshold *and* β is not too low. If β is too low, then the threshold is undefined and self 1 always prefers the equilibrium over commitment no matter how large the number of decision nodes. The proofs for all of these results are in Appendix A.

For example, suppose $c^- = 1/2$, $c^+ = 1$, $\beta = 0.4$, and $\delta = 0.85$. If so, then there must be 4 or more nodes in order for the holdup self (self 1) to prefer commitment over the equilibrium. Recall that when $\delta = 1$, we only needed 3 or more selves. For this particular δ, as long as $\beta > 0.176471$ then we can always increase the number of nodes to deliver the result that commitment multiself Pareto dominates the equilibrium.

3.4 A final example: when utility is not strictly increasing

In this subsection we show that when utility is not strictly increasing, the frequency of choice can have a fundamental effect on whether or not the problem is even time inconsistent in the first place.

Suppose the utility function is non-increasing in the following sense. Utility from unripe fruit is c^-, utility from ripe fruit is c^+, and utility from eating a piece of both types of fruit at once is c^+. That is, the individual becomes completely satiated after a piece of ripe fruit. Note that at a given point in time, one can eat an unripe piece of fruit, and/or a ripe piece if the previous self was patient. But there is no way to eat more than two pieces of fruit at once, nor is it possible to eat two pieces of unripe fruit or two pieces of ripe fruit. The most fruit that can be consumed at a single node is one unripe piece and one ripe piece.

In this setting the time-zero commitment allocation is still the same as before $\mathbf{c^0} = (0, c^+, ..., c^+)$. Note that the individual at time 1 can fully implement the commitment allocation by being patient himself, if he chooses to do so. If so, then self 2 will have a ripe piece of fruit and an unripe piece on the tree, and self 2 will choose to eat the ripe piece and leave the unripe piece until the next period, and so on. Hence, by simply acting patiently at the first node, the individual sets off a chain reaction such that he only eats ripe fruit, just as the time-zero self wishes. This is therefore a rather uninteresting case since there is no dynamic inconsistency (the commitment and equilibrium paths are the same). But there is no guarantee that self 1 will choose to be patient and set off this chain reaction. One needs to solve the sophisticated game by backward induction to determine which self, if any, will set off the chain reaction. After solving the game in this way, we uncover the following result.

Proposition 7 *Suppose* $\beta \in \left(\frac{c^-}{2c^+ - c^-}, \frac{c^-}{c^+}\right)$. *If the total number of decision nodes* T *is odd, then the choice problem is dynamically consistent,* $\mathbf{c^0} = \mathbf{c}^*$. *If* T *is even, then the choice problem is dynamically inconsistent and the time-zero commitment allocation cannot multiself Pareto dominate the equilibrium allocation.*

Intuitively, in this sophisticated game when T is even, self 1 knows that self 2 will break down and act patiently in order to constrain all future selves to do the same. Knowing this, he acts impatiently and lets self 2 do the hard work of abstaining. The resulting allocation always looks better to self 1 than the time-zero commitment allocation.

Dropping below the range of β defined in the above proposition alters the equilibrium of the sophisticated game, but not the spirit of the proposition.

Corollary 8 *Suppose* $\beta \in \left(\frac{c^-}{3c^+ - 2c^-}, \frac{c^-}{2c^+ - c^-}\right)$. *If* $T - 1$ *is divisible by 3, then the choice problem is dynamically consistent,* $\mathbf{c^0} = \mathbf{c}^*$. *Otherwise the choice problem is dynamically inconsistent and the time-zero commitment allocation cannot multiself Pareto dominate the equilibrium allocation.*

If $T - 1$ is not divisible by 3, then self 1 acts impatiently and games his later selves. He knows that either self 2 or self 3 will break down and set off the chain reaction of pre-commitment. Self 1 prefers either of these outcomes to the time-zero allocation, making the multiself Pareto result an impossible one to obtain.

4 Non-stationary examples

We now discuss a non-stationary setting where the choice set evolves over time. To be concrete in language, we describe the setting in terms of eating a cake, which is a work horse dynamic programming problem that is commonly used in a variety of contexts like saving for retirement. Essentially, our setting generalizes Hall's (1998) cake-eating model to many periods.

As in the fruit-tree example, our goal here is to show that when preferences are dynamically inconsistent and each self has a different perspective on what is optimal, it is possible to construct examples where the commitment allocation dominates the actual allocation from the perspective of all the selves. And in particular, we show that simply increasing the total number of decision nodes in the choice problem tends to bring about this result.

There is an infinitely divisible quantity of cake C. At $t = 0$ the individual orders a cake that will arrive at $t = 1$, and hence no cake can be eaten at $t = 0$ but cake may be eaten at all the other T decision nodes. The cake doesn't spoil (nor does it grow). The flow of consumption of the cake at node t is c_t. Any cake not yet consumed is available for consumption at future decision nodes. Self t applies the discount function $F(s)$ to future utility that is s years away, with $F(0) = 1$. For simplicity, period utility is log.[7]

In Appendix B we show that the commitment allocation is

$$c_t = \frac{CF(t)}{\sum_{s=1}^{T} F(s)}, \text{ for all } t > 0,$$

however the equilibrium allocation satisfies the following recursion

$$c_{t+1} = c_t \left(\frac{\sum_{s=1}^{T-t} F(s)}{1 + \sum_{s=1}^{T-t-1} F(s)} \right) < c_t.$$

4.1 β discounting

If the individual discounts all future utility by β, then the commitment allocation is

$$c_t = C/T, \text{ for all } t,$$

and the equilibrium allocation is

$$c_t = \left(\frac{C}{1 + \beta(T-1)} \right) \times \prod_{s=1}^{t-1} \left(\frac{\beta(T-s)}{1 + \beta(T-s-1)} \right), \text{ for } t > 0.$$

[7] Unlike the fruit tree example in which we had the luxury of working with linear utility, here we must assume utility is strictly concave. Without concavity, there is no unique solution to the time-zero cake-eating problem under once-off discounting $\{1, \beta, \beta, \beta, ...\}$ because the time-zero self doesn't care when the cake is eaten. And under quasi-hyperbolic discounting $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$ with linear utility, then there is a well-defined solution to the time-zero problem but it is the same as the equilibrium allocation: eat all the cake at $t = 1$, and in this case, commitment has nothing to offer.

We define the lifetime utility of self $t > 0$, over consumption allocation $\mathbf{c}$ with backward discount factor γ, as

$$U(t, \mathbf{c}) = \gamma \sum_{s=1}^{t-1} \ln c_s + \ln c_t + \beta \sum_{s=t+1}^{T} \ln c_s.$$

The threshold number of total decision nodes ranges from a low of 4 to a high of 9 for the rectangle defined by $\beta \in [0.2, 0.8]$ and $\gamma \in [0, 1]$. Any number of decision nodes above these threshold values leads to the conclusion that commitment multiself Pareto dominates the equilibrium. In other words, for any parameterization of β and γ within this space, 9 decision nodes is always sufficient to deliver our Pareto result, and in some cases just 4 nodes is enough. Simply expanding the number of opportunities that the individual has to eat a portion of the fixed quantity of cake (i.e., expanding the frequency of choice) brings the time-zero welfare criterion into alignment with a multiself Pareto criterion, just as in the fruit-tree example above.

4.2 $\beta\delta$ discounting

If the forward discount function is $\beta\delta^s$, for a delay of length s, then the commitment allocation is

$$c_t = \frac{C\delta^t}{\sum_{s=1}^{T} \delta^s} = C\delta^t \left(\frac{1-\delta}{\delta - \delta^{T+1}} \right), \text{ for all } t > 0,$$

and the equilibrium consumption allocation is

$$c_1 = \frac{C}{1 + \beta \sum_{s=1}^{T-1} \delta^s} = \frac{C(1-\delta)}{1 - \delta + \beta \left(\delta - \delta^T \right)}$$

$$c_{t+1} = c_t \left(\frac{\beta \sum_{s=1}^{T-t} \delta^s}{1 + \beta \sum_{s=1}^{T-t-1} \delta^s} \right) = \beta c_t \left(\frac{\delta - \delta^{T-t+1}}{1 - \delta + \beta \left(\delta - \delta^{T-t} \right)} \right), \text{ for } t > 0.$$

For self $t > 0$ with backward discount function $\gamma\eta^s$ for a delay of length s, a consumption allocation $\mathbf{c}$ confers utility

$$U(t, \mathbf{c}) = \gamma \sum_{s=1}^{t-1} \eta^{t-s} \ln c_s + \ln c_t + \beta \sum_{s=t+1}^{T} \delta^{s-t} \ln c_s.$$

We find that the principle lesson continues to go through: for a given parameterization (now there are four parameters instead of two, $\beta, \delta, \gamma, \eta$) there is a threshold number of decision nodes, below which the commitment allocation does not make all the selves better off relative to the equilibrium, and above which commitment multiself Pareto dominates the equilibrium.

For example, consider a fairly typical calibration of the quasi-hyperbolic forward discount function $\beta = 0.8, \delta = 0.95$, and suppose the backward discount function takes the same shape $\gamma = 0.8, \eta = 0.95$. In this case, as long as there are 4 or more decision nodes ($T \geq 4$) then the commitment allocation multiself Pareto dominates the equilibrium allocation. Suppose we keep the same assumptions about the forward discount function $\beta = 0.8, \delta = 0.95$, but now we assume the individual doesn't care at all about the past ($\gamma = 0$). As in the previous example, it still happens to be the case that as long as there are 4 or more decision nodes ($T \geq 4$) then the commitment allocation multiself Pareto dominates the equilibrium allocation.

5 Frequency of choice vs. total length of model time

In this final section we extend our analysis to continuous time, following in the tradition of Strotz (1955) and Pollak (1968).[8] We focus on a cake-eating example as in the previous section. Time is indexed by t. Time starts at $t = 0$ and ends at $t = T$.

Our purpose is to distinguish between the frequency of choice and the total length of model time. Time is always continuous and the total length of time is fixed at T. Decision nodes, however, are intentionally discrete so that we have the flexibility to adjust the frequency of choice while holding fixed the total length of model time.

The individual re-optimizes after Δ units of time have passed, and the individual is sophisticated in the sense that the current self knows that future selves will make different consumption choices than he would like them to make, and he takes this into account when he makes his current decisions. Hence, there are T/Δ different selves (decision nodes). Time is continuous between decision nodes. The decision nodes are $t = 0$, $t = \Delta$, $t = 2\Delta$,..., $t = T - 2\Delta$, $t = T - \Delta$. We are not referring to T as a decision node.

The stock of infinitely divisible cake is $k(t)$ and the individual consumes cake at rate $c(t)$, where $k(0) = 1$ and $k(T) = 0$. Period utility $u(c) = \ln c$, and one util that is s years away is discounted according to a discount function $F(s)$.

In Appendix C we show that the commitment allocation is

$$c^0(t) = \left(\int_0^T F(t)dt \right)^{-1} \times F(t), \text{ for } t \in [0, T],$$

[8] Barro (1999), Karp (2007), and Harris and Laibson (2011) also study a continuous time setting. However, they focus on the case where the time horizon is infinite, whereas we assume the horizon is finite in order to study a finite number of decision nodes.

and solving the model recursively gives the equilibrium allocation from the dynamic game

$$c^*(t) = k(0) \times \left(\int_0^T F(t)dt \right)^{-1} \times F(t), \text{ for } t \in [0, \Delta],$$

$$c^*(t) = k(\Delta) \times \left(\int_\Delta^T F(t-\Delta)dt \right)^{-1} \times F(t-\Delta), \text{ for } t \in [\Delta, 2\Delta],$$

$$c^*(t) = k(2\Delta) \times \left(\int_{2\Delta}^T F(t-2\Delta)dt \right)^{-1} \times F(t-2\Delta), \text{ for } t \in [2\Delta, 3\Delta],$$

and so on. The commitment allocation multiself Pareto dominates the equilibrium allocation if and only if

$$\int_t^T F(s-t)\ln[c^0(s)]ds > \int_t^T F(s-t)\ln[c^*(s)]ds \text{ for all } t.$$

Suppose we normalize the length of the time horizon to $T = 1$, and we use the continuous hyperbolic function $F(s) = (1 + \alpha s)^{-1}$ with $\alpha = 10$. Figure 4 plots the commitment allocation along with three equilibrium allocations for 5, 20, and 100 decision nodes as an example. Notice that the smaller the number of decision nodes, the more power a given self has to influence the equilibrium allocation. For instance, with only 5 nodes, the self in charge at $t = 0.2$ eats a huge slice of cake at that moment because he knows that he can commit all selves during the relatively large period of time leading up to the next decision node at $t = 0.4$. When there are more decision nodes, the influence of a given self becomes diffuse and we see smaller spikes in consumption at decision nodes.

At this particular parameterization, when there are just 5 decision nodes, commitment does not multiself Pareto dominate the equilibrium. The self at $t = 0.2$ prefers the equilibrium allocation because it confers so much more consumption at that moment than the commitment allocation. Other selves very close to $t = 0.2$ feel the same way. But as soon as we get to 10 nodes and beyond, the commitment allocation multiself Pareto dominates the equilibrium allocation. As the number of nodes gets larger, the self at any moment in time exerts very little influence on the equilibrium allocation and, while commitment is the first-best allocation only for self 0, all selves prefer it to the equilibrium.

6 Concluding remarks

We study a fundamental question in behavioral economics: How should scarce resources be allocated when individuals have dynamically inconsistent preferences? This is a difficult question to answer because the many time-dated selves of a single individual disagree on how resources should be allocated over time.

However, we show that the frequency of choice may play an important role in such welfare discussions. As the number of decision nodes increases (i.e., as the frequency of choice increases) the commitment allocation tends to multiself Pareto dominate the equilibrium allocation. With many selves, instead of just a few, the power to influence the equilibrium allocation becomes diffuse and the equilibrium therefore diverges from the ideal allocations of any of the selves. This gap opens the door for a Pareto improvement. While we do not claim that this finding will extend to all economic settings, it does persist across the set of important examples that we consider.

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7 Appendix A: proofs and derivations for fruit-tree example

7.1 Proof of Propositions 1 and 2

First consider the case of $T = 2$. The equilibrium allocation is $\mathbf{c}^* = (c^-, 0)$, which confers utility $U(1, \mathbf{c}^*) = c^-$ to self 1. On the other hand the commitment allocation $\mathbf{c^0} = (0, c^+)$ confers utility $U(1, \mathbf{c^0}) = \beta c^+$, which is less than $U(1, \mathbf{c}^*)$ because we have already assumed $\beta < c^-/c^+$. The equilibrium allocation dominates the commitment allocation from the perspective of self 1 and hence commitment cannot represent a multiself Pareto improvement (regardless of how self 2 feels about the commitment allocation over the equilibrium allocation).

However, adding just one more decision node overturns this result. Now suppose $T > 2$. An allocation $\mathbf{c}$ confers lifetime utility from the perspective of vantage point $t > 0$ according to

$$U(t, \mathbf{c}) = c_t + \beta \sum_{s=t+1}^{T} c_s \text{ for } t \in [1, T].$$

Self 1

$$U(1, \mathbf{c^0}) = 0 + \beta \sum_{s=2}^{T} c^+ = \beta(T-1)c^+$$

$$U(1, \mathbf{c}^*) = c^- + \beta \sum_{s=2}^{T-1} c^- = c^- + \beta(T-2)c^-.$$

He prefers commitment over the equilibrium, $U(1, \mathbf{c^0}) > U(1, \mathbf{c}^*)$, if

$$\beta(T-1)c^+ > c^- + \beta(T-2)c^-$$

or

$$\beta > \frac{c^-}{(T-2)(c^+ - c^-) + c^+} \equiv \bar{\beta}(T).$$

All of the later selves $t > 1$ are sure to prefer the commitment allocation over the equilibrium allocation for any β, so self 1 is the only potential holdup in drawing the conclusion that the commitment allocation multiself Pareto dominates the equilibrium allocation. Also recall that we assumed that $\beta < c^-/c^+$ so that there is enough discounting to ensure disagreement between the multiple selves (otherwise preferences are time consistent and all the selves choose to be patient, just like self 0 would like). Hence, to get time-inconsistent preferences and a Pareto role for commitment, we need

$$\beta \in \left(\bar{\beta}(T), \frac{c^-}{c^+}\right) \equiv S,$$

and S is non-empty as long as $T > 2$. This completes the proof of Proposition 1.

The proof of Proposition 2 follows immediately. The larger is T, the larger is the parameter space S that can deliver a Pareto gain from commitment, $d\bar{\beta}(T)/dT < 0$. In fact,

$$\lim_{T\to\infty} \bar{\beta}(T) = 0,$$

which means that the commitment allocation multiself Pareto dominates the equilibrium allocation for *any* $\beta < c^-/c^+$ if the tree lives forever.

7.2 Proof of Corollary 3

Inverting the condition $\beta > \bar{\beta}(T)$, while recognizing that T is an integer,

$$T \geq \bar{T}\left(2 + \frac{c^- - \beta c^+}{\beta(c^+ - c^-)}\right),$$

where we define $\bar{T}(x)$ as a mapping of real number x to the smallest integer that is greater than x. This condition ensures that the commitment allocation multiself Pareto dominates the equilibrium allocation. We know that $\bar{T} > 2$ because we have already assumed that the decision problem is dynamically inconsistent ($\beta < c^-/c^+$).

In sum, for any $\beta < c^-/c^+$, as long as the total number of decision nodes is equal to or exceeds $\bar{T}(x)$ then all selves will prefer commitment over the equilibrium. If there are fewer nodes than this threshold, then self 1 prefers the equilibrium.

7.3 Proof of Proposition 4

The lifetime utility of the individual at age t is

$$U(t, \mathbf{c}) = \begin{cases} \beta \sum_{s=1}^{T} c_s & t = 0 \\ \gamma \sum_{s=1}^{t-1} c_s + c_t + \beta \sum_{s=t+1}^{T} c_s & t \in [1, T]. \end{cases}$$

Previously, we focused on the special case $\gamma = 0$. The purpose of this proposition and proof is to illustrate that we have not lost any generality in doing so, though the more general case is more cumbersome algebraically. All the same results hold when we allow for any value of $\gamma < 1$: the commitment allocation Pareto dominates the equilibrium allocation if and only if self 1 prefers commitment, and this happens if $\beta > \bar{\beta}(T)$ as in Figure 1.

Consider self 1

$$U(1, \mathbf{c^0}) = 0 + \beta \sum_{s=2}^{T} c^+ = \beta(T-1)c^+$$

$$U(1, \mathbf{c}^*) = c^- + \beta \sum_{s=2}^{T-1} c^- = c^- + \beta(T-2)c^-.$$

He prefers commitment over the equilibrium, $U(1, \mathbf{c^0}) > U(1, \mathbf{c}^*)$, if

$$\beta > \frac{c^-}{(T-2)(c^+ - c^-) + c^+} \equiv \bar{\beta}(T).$$

For self $t \in [2, T-1]$

$$U(t, \mathbf{c^0}) = \gamma \sum_{s=2}^{t-1} c^+ + c^+ + \beta \sum_{s=t+1}^{T} c^+ = \gamma(t-2)c^+ + c^+ + \beta(T-t)c^+$$

$$U(t, \mathbf{c}^*) = \gamma \sum_{s=1}^{t-1} c^- + c^- + \beta \sum_{s=t+1}^{T-1} c^- = \gamma(t-1)c^- + c^- + \beta(T-t-1)c^-.$$

And hence

$$U(t, \mathbf{c^0}) > U(t, \mathbf{c}^*) \text{ for all } t \in [2, T-1]$$

if

$$\beta > \frac{\gamma\left[(t-1)c^- - (t-2)c^+\right] + c^- - c^+}{(T-t)c^+ - (T-t-1)c^-} \equiv \hat{\beta}(\gamma, T, t) \text{ for all } t \in [2, T-1].$$

Self T

$$U(T, \mathbf{c^0}) = c^+ + \gamma(T-2)c^+$$

$$U(T, \mathbf{c}^*) = \gamma(T-1)c^- = \gamma c^- + \gamma(T-2)c^-$$

and hence it is always the case that

$$U(T, \mathbf{c^0}) > U(T, \mathbf{c}^*).$$

In summary, for a given number of nodes T, the commitment allocation multiself Pareto dominates the equilibrium allocation from the perspective of self 1 if

$$\beta \in \left(\bar{\beta}(T), \frac{c^-}{c^+}\right).$$

Likewise, for a given number of nodes T, and for a given backward discount factor γ, the commitment allocation dominates the equilibrium allocation from the perspective of self $t \in [2, T-1]$ if

$$\beta \in \left(\hat{\beta}(\gamma, T, t), \frac{c^-}{c^+}\right).$$

However, it can be show that

$$\hat{\beta}(\gamma, T, t) < \bar{\beta}(T) \text{ for all } t \in [2, T-1] \text{ and for all } \gamma < 1$$

which means that if the commitment allocation dominates the equilibrium allocation from the perspective of self 1, then the commitment allocation will multiself Pareto dominate the equilibrium allocation.

To see this, note that

$$\frac{\partial \hat{\beta}(\gamma, T, t)}{\partial t} = (c^- - c^+) \times \left(\frac{\gamma(T-2)\,(c^+ - c^-) + c^+ - c^-}{[(T-t)c^+ - (T-t-1)c^-]^2}\right) < 0 \text{ for all } t \geq 2$$

which means that

$$2 = \arg\max_t \hat{\beta}(\gamma, T, t)$$

$$\hat{\beta}(\gamma, T, 2) = \frac{\gamma c^- + c^- - c^+}{(T-2)c^+ - (T-3)c^-}.$$

Further note that $\hat{\beta}(\gamma, T, 2)$ is maximized in the γ dimension as $\gamma \to 1$ (we are assuming γ cannot exceed 1). Rewrite

$$\bar{\beta}(T) = \frac{c^-}{(T-2)(c^+ - c^-) + c^+}$$

$$\hat{\beta}(1, T, 2) = \frac{2c^- - c^+}{(T-2)(c^+ - c^-) + c^-}$$

and compute

$$\bar{\beta}(T) - \hat{\beta}(1, T, 2) = \frac{c^-[(T-2)(c^+ - c^-) + c^-] - (2c^- - c^+)[(T-2)(c^+ - c^-) + c^+]}{[(T-2)(c^+ - c^-) + c^+][(T-2)(c^+ - c^-) + c^-]}.$$

The denominator is positive, and with some algebra the numerator can be rewritten as

$$\left(c^{+}-c^{-}\right)^{2}(T-1)>0$$

and hence

$$\bar{\beta}(T)>\hat{\beta}(1,T,2)>\hat{\beta}(\gamma,T,t),\ \text{ for all } t\in[2,T-1] \text{ and for all } \gamma<1.$$

Therefore, as in the previous special case where $\gamma=0$, self 1 is the holdup. The commitment allocation Pareto dominates the equilibrium allocation if and only if self 1 prefers commitment, and this happens if $\beta>\bar{\beta}(T)$ as in Figure 1, and therefore all the lessons and intuition from the special case $\gamma=0$ carries over to the case of generic γ.

7.4 Proof of Proposition 5

Suppose time starts at $t=1$ and there is already the option to consume (there already exists an unripe piece of fruit sitting on the tree). Now we will refer to $\mathbf{c^1}$ as the commitment allocation, and assuming $\beta<c^{-}/c^{+}$ (time inconsistency) we have

$$\mathbf{c^1}=(c^{-},0,c^{+},...,c^{+}),\quad \mathbf{c}^{*}=(c^{-},...,c^{-},0).$$

Consider self 2

$$U(2,\mathbf{c^1})=0+\beta\sum_{s=3}^{T}c^{+}=\beta(T-2)c^{+}$$

$$U(2,\mathbf{c}^{*})=c^{-}+\beta\sum_{s=3}^{T-1}c^{-}=c^{-}+\beta(T-3)c^{-}.$$

He prefers commitment over the equilibrium, $U(2,\mathbf{c^1})>U(2,\mathbf{c}^{*})$, if

$$\beta(T-2)c^{+}>c^{-}+\beta(T-3)c^{-}$$

or

$$\beta>\frac{c^{-}}{(T-3)(c^{+}-c^{-})+c^{+}}.$$

All of the later selves $t>2$ prefer commitment over the equilibrium for any β, so self 2 is the only potential holdup in drawing the conclusion that the commitment allocation $\mathbf{c^1}$ multiself Pareto dominates the equilibrium allocation. Hence, to get time-inconsistent preferences and a Pareto role for commitment, we need

$$\beta \in \left(\frac{c^-}{(T-3)(c^+ - c^-) + c^+}, \frac{c^-}{c^+} \right) \equiv S,$$

and S is non-empty as long as $T > 3$.

7.5 Proof of Proposition 6

First consider all potential self-t commitment allocations for $t > 1$. In this case, self-t would prefer the self-0 commitment allocation over the self-t commitment allocation, the next self $t+1$ would prefer the same, and all selves after that would be indifferent. Likewise, all selves before self t (except the first self) would unequivocally prefer the self-0 commitment allocation over the self-t commitment allocation. The first self would prefer the self-0 commitment allocation over the self-t commitment allocation if

$$\beta \in \left(\frac{c^-}{(t-1)(c^+ - c^-) + c^+}, \frac{c^-}{c^+} \right),$$

which is non-empty for $t > 1$. The lower bound of this set is largest when t is smallest, so setting $t = 2$ we conclude that if

$$\beta \in \left(\frac{c^-}{2c^+ - c^-}, \frac{c^-}{c^+} \right),$$

then the self-0 commitment allocation multiself Pareto dominates all other self-t commitment allocations for $t > 1$.

Now consider the self-1 commitment allocation. Self 2 would prefer the self-0 commitment allocation over the self-1 commitment allocation, and selves 3 and beyond would be indifferent. Self 1 however, would never prefer the self-0 commitment allocation over the self-1 commitment allocation, because the later is ideal for him.

7.6 Proof of claims in $\beta\delta$ discounting subsection

Suppose $T > 2$. Consider self 1

$$U(1, \mathbf{c^0}) = 0 + \beta \sum_{s=2}^{T} \delta^{s-1} c^+$$

$$U(1, \mathbf{c}^*) = c^- + \beta \sum_{s=2}^{T-1} \delta^{s-1} c^-.$$

He prefers commitment over the equilibrium, $U(1,\mathbf{c^0}) > U(1,\mathbf{c}^*)$, if

$$\beta \sum_{s=2}^{T} \delta^{s-1} c^+ > c^- + \beta \sum_{s=2}^{T-1} \delta^{s-1} c^-$$

or

$$\frac{\beta(c^+ - c^-)}{\delta} \sum_{s=2}^{T-1} \delta^s + \beta\delta^{T-1}c^+ > c^-.$$

Recalling the sum of a finite geometric series we get

$$\delta + \delta^T + \sum_{s=2}^{T-1} \delta^s = \sum_{s=1}^{T} \delta^s = \frac{\delta(1-\delta^T)}{1-\delta}$$

and hence

$$\frac{\beta(c^+ - c^-)}{\delta} \left(\frac{\delta(1-\delta^T)}{1-\delta} - \delta - \delta^T \right) + \beta\delta^{T-1}c^+ > c^-.$$

Rewrite as

$$\frac{\beta(c^+ - c^-)}{\delta(1-\delta)} (\delta^2 - \delta^T) + \beta\delta^{T-1}c^+ > c^-,$$

and note that $\delta^2 - \delta^T > 0$ when $T > 2$. Solve for β

$$\beta > \frac{c^-}{(c^+ - c^-)(\delta^2 - \delta^T)/(\delta - \delta^2) + \delta^{T-1}c^+} \equiv \bar{\beta}(T).$$

Hence, commitment strictly multiself Pareto dominates the equilibrium if

$$\beta \in \left(\bar{\beta}(T), \frac{c^-}{\delta c^+} \right) \equiv S.$$

Note that S is empty at $T = 2$. Using the restriction $\delta > c^-/c^+$ we can sign the following derivative

$$\frac{d}{dT}[(c^+ - c^-)(\delta^2 - \delta^T)/(\delta - \delta^2) + \delta^{T-1}c^+] = \left(\frac{c^- - \delta c^+}{1-\delta} \right) \delta^{T-1} \ln \delta > 0 \Longrightarrow \frac{d\bar{\beta}(T)}{dT} < 0.$$

Hence, S is non-empty for all $T > 2$. In other words, for any $T > 2$, we can *always* find a range of β values for which commitment multiself Pareto dominates the equilibrium.

Set S is monotonically increasing in T because the lower bound of the set is monotonically decreasing in T and the upper bound is fixed. However, unlike in the simple case of β discounting where the lower bound limits to zero, here the lower bound limits to a value greater than zero,

$$\lim_{T\to\infty} \bar{\beta}(T) = \frac{(1-\delta)c^-}{\delta(c^+ - c^-)} > 0.$$

Finally, rather than solving for β, let us rewrite the condition for a Pareto improvement as

$$\beta \left(\frac{c^- - \delta c^+}{1-\delta} \right) \delta^{T-1} > c^- - \frac{\beta(c^+ - c^-)}{(1-\delta)} \delta.$$

Solve for T (making use of the restriction $\delta > c^-/c^+$ which implies $\beta\left(\frac{c^- - \delta c^+}{1-\delta}\right) < 0$ and also that $\ln \delta < 0$)

$$T \geq \bar{T}\left(1 + \frac{1}{\ln \delta} \ln \left(\frac{\beta(c^+ - c^-)\delta - (1-\delta)c^-}{\beta\left(\delta c^+ - c^-\right)} \right)\right),$$

where $\bar{T}(x)$ is a mapping of real number x to the smallest integer that is greater than x. If $T \geq \bar{T}(x)$ then commitment multiself Pareto dominates the equilibrium, otherwise self 1 prefers the equilibrium to commitment. Note that if $\beta < \lim_{T\to\infty} \bar{\beta}(T)$, then the argument in the second log term in $\bar{T}(x)$ is negative, which means that the threshold itself is undefined.

7.7 Proof of Proposition 7

We obtain the equilibrium consumption path to the sophisticated game by backward induction. One key point to recall is that a given self t will choose to be patient if self $t-1$ was patient, which sets off a chain reaction such that all selves t and beyond are patient if $t-1$ is patient. This is because self t has two pieces of fruit at his disposal, and he will eat the ripe one and only the ripe one because it cannot be stored an extra period and eating an unripe piece in addition to the ripe one doesn't add any utility and the same process repeats itself.

Start with self $T-1$. Because $\beta < c^-/c^+$ by assumption, self $T-1$ will be *impatient* if he has not already been "pre-committed", that is, if no earlier self set off the chain reaction of acting patiently.

Self $T-2$ knows that self $T-1$ will be impatient if $T-1$ is not already pre-committed. Given the restriction $\beta > c^-/(2c^+ - c^-)$, we have $(0, c^+, c^+) \succ (c^-, c^-, 0)$ and self $T-2$ chooses to set off the chain reaction of acting *patiently* if this chain reaction has not already been triggered.

Self $T-3$ knows that self $T-2$ will be patient irrespective of the choice of self $T-3$. If self $T-3$ is not already pre-committed, he will choose to be *impatient* now and let self $T-2$ set off the chain reaction. This is seen by noting that $\beta < c^-/c^+$ implies that $(c^-, 0, c^+, c^+) \succ (0, c^+, c^+, c^+)$.

Self $T-4$ knows that self $T-3$ will be impatient if $T-3$ is not already pre-committed, and he further knows that self $T-2$ will choose to set off the chain reaction of acting patiently. Given the restriction $\beta > c^-/(2c^+ - c^-)$, we have $(0, c^+, c^+, c^+, c^+) \succ (c^-, c^-, 0, c^+, c^+)$ and self $T-4$ chooses to set off the chain reaction now (if it has not already been set off) by acting *patiently* rather than waiting for self $T-2$ to set off the reaction.

Self $T-5$ knows that self $T-4$ will be patient irrespective of the choice of self $T-5$. If self $T-5$ is not already pre-committed, he will choose to be *impatient* now and let self $T-4$ set off the chain reaction. This is seen by noting that $\beta < c^-/c^+$ implies that $(c^-, 0, c^+, c^+, c^+, c^+) \succ (0, c^+, c^+, c^+, c^+, c^+)$.

Self $T-6$ knows that self $T-5$ will be impatient if $T-5$ is not already pre-committed, and he further knows that self $T-4$ will choose to set off the chain reaction of acting patiently. Given the restriction $\beta > c^-/(2c^+ - c^-)$, we have $(0, c^+, c^+, c^+, c^+, c^+, c^+) \succ (c^-, c^-, 0, c^+, c^+, c^+, c^+)$ and self $T-6$ chooses to set off the chain reaction now (if it has not already been set off) by acting *patiently* rather than waiting for self $T-4$ to set off the reaction.

Noting the pattern, if i is odd then self $T-i$ will be impatient if he is not already pre-committed. Likewise, if i is even then self $T-i$ will be patient and will start the chain reaction if it has not already been triggered.

At $i = T-1$, the individual is standing at $t = T-(T-1) = 1$, so if $T-1$ is even (meaning T is odd) then self 1 acts patiently and pre-commits all future selves, which is just what self 0 wants him to do. In this case the problem is dynamically consistent, $\mathbf{c^0} = \mathbf{c}^* = (0, c^+, ..., c^+)$. If on the other hand $T-1$ is odd (meaning T is even) then self 1 acts impatiently, knowing that self 2 will pre-commit all future selves by acting patiently. In this case the problem is dynamically inconsistent, $\mathbf{c^0} = (0, c^+, ..., c^+)$ and $\mathbf{c}^* = (c^-, 0, c^+, ..., c^+)$, and self 1 will never prefer $\mathbf{c^0}$ over $\mathbf{c}^*$ because $\mathbf{c}^*$ is his ideal allocation. Hence, all selves do not prefer commitment over the equilibrium.

7.8 Proof of Corollary 8

Recall that a given self t will choose to be patient if self $t-1$ was patient, which sets off a chain reaction such that all selves t and beyond are patient if $t-1$ is patient. So finding the equilibrium of the game is a matter of identifying which self will set off the chain reaction, and we find this information by backward induction.

Start with self $T-1$. Because $\beta < c^-/c^+$ by assumption, self $T-1$ will be *impatient* if he has not already been "pre-committed", that is, if no earlier self set off the chain reaction of acting patiently.

Self $T-2$ knows that self $T-1$ will be impatient if $T-1$ is not already pre-committed. Given the restriction $\beta < c^-/(2c^+ - c^-)$, we have $(c^-, c^-, 0) \succ (0, c^+, c^+)$ and self $T-2$ chooses to be *impatient* as well, if no self before him has set off the chain reaction of acting patiently.

Self $T-3$ knows that selves $T-2$ and $T-1$ will be impatient if not already pre-committed. Whether or not self $T-3$ is already pre-committed, he will choose to be *patient* now because $\beta > c^-/(3c^+ - 2c^-)$ implies that $(0, c^+, c^+, c^+) \succ (c^-, c^-, c^-, 0)$.

Self $T-4$ knows that self $T-3$ will be patient either way, so if $T-4$ is not already pre-committed then he chooses to be *impatient* because $\beta < c^-/c^+$ implies $(c^-, 0, c^+, c^+, c^+) \succ (0, c^+, c^+, c^+, c^+)$ and in order to let self $T-3$ set off the chain reaction a period later.

Self $T-5$ knows that self $T-4$ will be impatient if not already pre-committed and that self $T-3$ will commit. If self $T-5$ is not already pre-committed, he will also choose to be *impatient* now and let self $T-3$ set off the chain reaction because $\beta < c^-/(2c^+ - c^-)$ implies $(c^-, c^-, 0, c^+, c^+, c^+) \succ (0, c^+, c^+, c^+, c^+, c^+)$.

Self $T-6$ knows that self $T-5$ will be impatient if $T-5$ is not already pre-committed, and he further knows that self $T-4$ will be impatient as well and that $T-3$ will choose to pre-commit remaining selves. Given the restriction $\beta > c^-/(3c^+ - 2c^-)$, we have $(0, c^+, c^+, c^+, c^+, c^+, c^+) \succ (c^-, c^-, c^-, 0, c^+, c^+, c^+)$ and self $T-6$ chooses to set off the chain reaction now by acting *patiently* rather than waiting for self $T-3$ to set off the reaction.

Iterating back even further confirms the following pattern. If i is divisible by 3, then self $T-i$ will be patient. Otherwise self $T-i$ will be impatient.

Suppose $i = T-1$ is divisible by 3. If so, then self $T-(T-1) = 1$ will be patient and implement the time-zero allocation. In this case the problem is dynamically consistent, $\mathbf{c^0} = \mathbf{c}^* = (0, c^+, ..., c^+)$.

If $i = T-2$ is divisible by 3, then self $T-(T-1) = 1$ knows that self $T-(T-2) = 2$ will be patient and self 1 strategically acts impatiently as a result. Self 2 implements the first-best allocation from the perspective of self 1, and hence self 1 would never prefer time-zero commitment to the equilibrium.

If $i = T-3$ is divisible by 3, then self $T-(T-3) = 3$ will be patient. Self $T-(T-1) = 1$ will choose impatience knowing that self 2 will do likewise and knowing that self 3 will commit. From the perspective of self 1, $\mathbf{c}^* = (c^-, c^-, 0, c^+, ..., c^+) \succ (0, c^+, ..., c^+) = \mathbf{c^0}$ because $\beta < c^-/(2c^+ - c^-)$ and again self 1 prefers the equilibrium over commitment.

8 Appendix B: derivation of cake-eating solution

Here we report analytical derivations of the commitment and equilibrium allocations for a cake-eating problem.

8.1 Commitment

Self 0 would like his future selves to obey

$$\max : \sum_{t=1}^{T} F(t) \ln c_t, \quad \text{s.t.} \ \sum_{t=1}^{T} c_t = C,$$

which has the following solution (commitment allocation)

$$c_t = \frac{CF(t)}{\sum_{s=1}^{T} F(s)}, \text{ for all } t > 0.$$

8.2 Equilibrium

Recalling that the individual is sophisticated, we solve the problem recursively:

At vantage point $T-1$ he solves

$$\max : \ln c_{T-1} + F(1) \ln c_T, \quad \text{s.t.} \ c_{T-1} + c_T = C - \sum_{t=1}^{T-2} c_t.$$

The first order condition is

$$c_T = F(1) c_{T-1},$$

which implies

$$c_{T-1} = \frac{C - \sum_{t=1}^{T-2} c_t}{1 + F(1)}.$$

At vantage point $T-2$ he solves

$$\max : \ln c_{T-2} + F(1) \ln c_{T-1} + F(2) \ln c_T$$

subject to[9]

$$\begin{aligned} c_{T-2}+c_{T-1}+c_T &= C-\sum_{t=1}^{T-3}c_t \\ c_{T-1} &= \frac{C-\sum_{t=1}^{T-2}c_t}{1+F(1)} \\ c_T &= F(1)c_{T-1} \end{aligned}$$

or

$$\max:\ln c_{T-2}+F(1)\ln\left(\frac{C-\sum_{t=1}^{T-2}c_t}{1+F(1)}\right)+F(2)\ln\left(F(1)\frac{C-\sum_{t=1}^{T-2}c_t}{1+F(1)}\right)$$

which is the same as maximizing

$$\max:\ln c_{T-2}+(F(1)+F(2))\ln\left(C-\sum_{t=1}^{T-3}c_t-c_{T-2}\right).$$

The first order condition is

$$\frac{1}{c_{T-2}}-\frac{F(1)+F(2)}{C-\sum_{t=1}^{T-3}c_t-c_{T-2}}=0$$

$$c_{T-2}=\frac{C-\sum_{t=1}^{T-3}c_t}{1+F(1)+F(2)}.$$

At vantage point $T-3$ he solves

$$\max:\ln c_{T-3}+F(1)\ln c_{T-2}+F(2)\ln c_{T-1}+F(3)\ln c_T$$

subject to[10]

$$\begin{aligned} c_{T-3}+c_{T-2}+c_{T-1}+c_T &= C-\sum_{t=1}^{T-4}c_t \\ c_{T-2} &= \frac{C-\sum_{t=1}^{T-3}c_t}{1+F(1)+F(2)} \\ c_{T-1} &= \frac{C-\sum_{t=1}^{T-2}c_t}{1+F(1)} \\ c_T &= F(1)c_{T-1}. \end{aligned}$$

[9] The first of these constraints—remaining consumption must equal remaining cake—is actually redundant because this constraint was already imposed in the first step of the recursive solution.

[10] Again, the first constraint is redundant and can be dropped.

Note that

$$c_{T-2} = \frac{C - \sum_{t=1}^{T-3} c_t}{1 + F(1) + F(2)} = \frac{C - \sum_{t=1}^{T-4} c_t - c_{T-3}}{1 + F(1) + F(2)}$$

$$\begin{aligned}
c_{T-1} &= \frac{C - \sum_{t=1}^{T-2} c_t}{1 + F(1)} = \frac{C - \sum_{t=1}^{T-4} c_t - c_{T-3} - c_{T-2}}{1 + F(1)} \\
&= \frac{C - \sum_{t=1}^{T-4} c_t - c_{T-3}}{1 + F(1)} - \frac{1}{1 + F(1)} \frac{C - \sum_{t=1}^{T-4} c_t - c_{T-3}}{1 + F(1) + F(2)} \\
&= \left(C - \sum_{t=1}^{T-4} c_t - c_{T-3} \right) \left(\frac{1}{1 + F(1)} \right) \left(\frac{F(1) + F(2)}{1 + F(1) + F(2)} \right).
\end{aligned}$$

Hence

$$\begin{aligned}
\max \quad : \quad & \ln c_{T-3} + F(1) \ln \left(\frac{C - \sum_{t=1}^{T-4} c_t - c_{T-3}}{1 + F(1) + F(2)} \right) \\
& + F(2) \ln \left(\left(C - \sum_{t=1}^{T-4} c_t - c_{T-3} \right) \left(\frac{1}{1 + F(1)} \right) \left(\frac{F(1) + F(2)}{1 + F(1) + F(2)} \right) \right) \\
& + F(3) \ln \left(F(1) \left(C - \sum_{t=1}^{T-4} c_t - c_{T-3} \right) \left(\frac{1}{1 + F(1)} \right) \left(\frac{F(1) + F(2)}{1 + F(1) + F(2)} \right) \right)
\end{aligned}$$

which is the same as maximizing

$$\max : \ln c_{T-3} + (F(1) + F(2) + F(3)) \ln \left(C - \sum_{t=1}^{T-4} c_t - c_{T-3} \right)$$

which has the first order condition

$$\frac{1}{c_{T-3}} - \frac{F(1) + F(2) + F(3)}{C - \sum_{t=1}^{T-4} c_t - c_{T-3}} = 0$$

which implies

$$c_{T-3} = \frac{C - \sum_{t=1}^{T-4} c_t}{1 + F(1) + F(2) + F(3)}.$$

Note the pattern that has emerged, standing n nodes back from the final node

$$c_{T-n} = \frac{C - \sum_{t=1}^{T-(n+1)} c_t}{1 + \sum_{t=1}^{n} F(t)}.$$

For convenience, change the index dummy from t to s

$$c_{T-n} = \frac{C - \sum_{s=1}^{T-(n+1)} c_s}{1 + \sum_{s=1}^{n} F(s)}$$

and note that standing n nodes back from the last node is the same as standing at node $t = T - n$, so

$$c_t = \frac{C - \sum_{s=1}^{t-1} c_s}{1 + \sum_{s=1}^{T-t} F(s)}, \text{ for } t > 0,$$

and hence we have the following recursion

$$\begin{aligned}
c_{t+1} &= \frac{C - \sum_{s=1}^{t} c_s}{1 + \sum_{s=1}^{T-t-1} F(s)} \\
&= \frac{C - \sum_{s=1}^{t-1} c_s - c_t}{1 + \sum_{s=1}^{T-t-1} F(s)} \\
&= \frac{C - \sum_{s=1}^{t-1} c_s}{1 + \sum_{s=1}^{T-t-1} F(s)} - \frac{c_t}{1 + \sum_{s=1}^{T-t-1} F(s)} \\
&= c_t \frac{1 + \sum_{s=1}^{T-t} F(s)}{1 + \sum_{s=1}^{T-t-1} F(s)} - \frac{c_t}{1 + \sum_{s=1}^{T-t-1} F(s)} \\
&= c_t \left(\frac{\sum_{s=1}^{T-t} F(s)}{1 + \sum_{s=1}^{T-t-1} F(s)} \right) \\
&< c_t.
\end{aligned}$$

9 Appendix C: equilibrium in continuous time

The commitment allocation solves

$$\max : \int_0^T F(t) \ln c(t) dt,$$

subject to

$$\frac{dk(t)}{dt} = -c(t), \text{ with } k(0) = 1 \text{ and } k(T) = 0,$$

which gives

$$c(t) = \left(\int_0^T F(t) dt \right)^{-1} \times F(t).$$

However, the equilibrium path that is actually followed is the solution to a dynamic game. We solve the game recursively, starting with the oldest self, and we derive in closed form the equilibrium consumption allocation.[11] We refer to the oldest self as "self Δ", since he is the one who controls decision making over the final interval of time $t \in [T - \Delta, T]$. The second-oldest self, "self 2Δ", controls decision making over the interval $t \in [T - 2\Delta, T - \Delta]$, and so on.

Our approach differs from the traditional method for solving a continuous time problem (Pollak (1968)). He solves the problem in two stages. First he solves a discrete time model to obtain consumption allocations at the decision nodes, and then he solves the problem of smoothing consumption between nodes. On the other hand, we solve the continuous time problem directly without resorting to discrete time optimization. We start with the oldest self and then recursively link the selves together through transversality conditions.

9.1 Problem Δ: oldest self

The oldest self is standing at age $t = T - \Delta$ and solves a *fixed* endpoint optimal control problem over the interval $[T - \Delta, T]$

$$\max : J(\Delta) = \int_{T-\Delta}^T F(t - (T - \Delta)) \ln c(t) dt, \tag{1}$$

subject

$$\frac{dk(t)}{dt} = -c(t), \tag{2}$$

$$k(T - \Delta) \text{ given}, \tag{3}$$

[11] Unlike the infinite horizon setting which suffers from a multiplicity of equilibria (Kocherlakota (1996); Krusell and Smith (2003, 2008); Karp (2007); Vieille and Weibull (2009)), the equilibrium to the dynamic game in this finite horizon setting is unique (Kocherlakota (1996)).

$$k(T) = 0. \tag{4}$$

The first order necessary conditions are

$$F(t-(T-\Delta))\frac{1}{c(t)} - \lambda(t) = 0, \tag{5}$$

$$\frac{d\lambda(t)}{dt} = 0. \tag{6}$$

Hence $\lambda(t) = \lambda$ is constant and (5) and (6) become

$$c(t) = \frac{1}{\lambda}F(t-(T-\Delta)). \tag{7}$$

Solving (2) with its boundary condition $k(T-\Delta)$ we have

$$k(t) = k(T-\Delta) - \int_{T-\Delta}^{t} c(z)dz. \tag{8}$$

Combine (7) and (8)

$$k(t) = k(T-\Delta) - \frac{1}{\lambda}\int_{T-\Delta}^{t} F(z-(T-\Delta))dz, \tag{9}$$

and evaluate (9) at $t = T$ and use $k(T) = 0$ to solve for $1/\lambda$

$$\frac{1}{\lambda} = k(T-\Delta) \times \left(\int_{T-\Delta}^{T} F(t-(T-\Delta))dt\right)^{-1}. \tag{10}$$

Insert (10) into (7) to obtain the solution consumption path for the interval $[T-\Delta, T]$

$$c^{\Delta}(t) = k(T-\Delta) \times \left(\int_{T-\Delta}^{T} F(t-(T-\Delta))dt\right)^{-1} \times F(t-(T-\Delta)). \tag{11}$$

9.2 Problem 2Δ: second-oldest self

The second-oldest self is standing at age $t = T - 2\Delta$ and solves a *free* endpoint optimal control problem over the interval $[T-2\Delta, T-\Delta]$, and he knows the oldest self will have control over decisions on the last interval $[T-\Delta, T]$ and will choose $c(t) = c^{\Delta}(t)$ given the savings $k(T-\Delta)$ that the oldest self inherits. Knowing this, the second-oldest self solves

$$\max : J(2\Delta) = \int_{T-2\Delta}^{T-\Delta} F(t-(T-2\Delta)) \ln c(t)dt + \mathcal{S}(k(T-\Delta)), \tag{12}$$

where $\mathcal{S}(k(T-\Delta))$ is the scrap value that he attaches to the savings account $k(T-\Delta)$ that he leaves to the oldest self, knowing exactly what the oldest self will do with the inheritance,

$$\begin{aligned}\mathcal{S}(k(T-\Delta)) &= \int_{T-\Delta}^{T} F(t-(T-2\Delta)) \ln c^{\Delta}(t) dt \\ &= \int_{T-\Delta}^{T} F(t-(T-2\Delta)) \ln \left[\frac{k(T-\Delta)F(t-(T-\Delta))}{\int_{T-\Delta}^{T} F(t-(T-\Delta))dt} \right] dt, \end{aligned} \tag{13}$$

subject to the constraints

$$\frac{dk(t)}{dt} = -c(t), \tag{14}$$

$$k(T-2\Delta) \text{ given}, \tag{15}$$

$$k(T-\Delta) \text{ free}. \tag{16}$$

The first order necessary conditions are

$$F(t-(T-2\Delta))\frac{1}{c(t)} - \lambda(t) = 0, \tag{17}$$

$$\frac{d\lambda(t)}{dt} = 0, \tag{18}$$

$$\lambda(T-\Delta) = \mathcal{S}'(k(T-\Delta)). \tag{19}$$

From (18) we see that $\lambda(t)$ is constant and hence combining this result with (19) we have

$$\lambda(t) = \lambda(T-\Delta) = \mathcal{S}'(k(T-\Delta)), \tag{20}$$

and hence from (17) we have

$$c(t) = F(t-(T-2\Delta))\frac{1}{\mathcal{S}'(k(T-\Delta))}. \tag{21}$$

Compute $\mathcal{S}'(k(T-\Delta))$

$$\mathcal{S}'(k(T-\Delta)) = k(T-\Delta)^{-1} \int_{T-\Delta}^{T} F(t-(T-2\Delta))dt. \tag{22}$$

Solving (14) with its boundary condition $k(T-2\Delta)$ we have

$$k(t) = k(T-2\Delta) - \int_{T-2\Delta}^{t} c(z)dz. \tag{23}$$

Evaluate (23) at $t = T - \Delta$

$$k(T-\Delta) = k(T-2\Delta) - \int_{T-2\Delta}^{T-\Delta} c(t)dt. \tag{24}$$

Insert (21) into (24)

$$k(T-\Delta) = k(T-2\Delta) - \int_{T-2\Delta}^{T-\Delta} F(t-(T-2\Delta))\frac{1}{\mathcal{S}'(k(T-\Delta))}dt, \tag{25}$$

and then insert (22) into (25)

$$k(T-\Delta) = k(T-2\Delta) - \frac{k(T-\Delta)}{\int_{T-\Delta}^{T} F(t-(T-2\Delta))dt}\int_{T-2\Delta}^{T-\Delta} F(t-(T-2\Delta))dt. \tag{26}$$

Now solve (26) for $k(T-\Delta)$

$$k(T-\Delta) = k(T-2\Delta)\frac{\int_{T-\Delta}^{T} F(t-(T-2\Delta))dt}{\int_{T-2\Delta}^{T} F(t-(T-2\Delta))dt}. \tag{27}$$

Finally, combine (22) and (27), and after some algebra the result is

$$\mathcal{S}'(k(T-\Delta)) = k(T-2\Delta)^{-1}\int_{T-2\Delta}^{T} F(t-(T-2\Delta))dt, \tag{28}$$

and then insert (28) into (21)

$$c^{2\Delta}(t) = k(T-2\Delta) \times \left(\int_{T-2\Delta}^{T} F(t-(T-2\Delta))dt\right)^{-1} \times F(t-(T-2\Delta)). \tag{29}$$

9.3 Problem 3Δ: third-oldest self

The third-oldest self is standing at age $t = T - 3\Delta$ and solves a *free* endpoint optimal control problem over the interval $[T-3\Delta, T-2\Delta]$, and he knows the second-oldest self will have control over the interval $[T-2\Delta, T-\Delta]$ and the oldest self will have control over the last interval $[T-\Delta, T]$. The third-oldest self knows the second-oldest self will choose $c(t) = c^{2\Delta}(t)$ given the savings $k(T-2\Delta)$ that the second-oldest self inherits, and he also knows that the oldest self will choose $c(t) = c^{\Delta}(t)$ given the savings $k(T-\Delta)$

that the oldest self inherits. Knowing all of this, the third-oldest self solves

$$\max : J(3\Delta) = \int_{T-3\Delta}^{T-2\Delta} F(t-(T-3\Delta)) \ln c(t) dt + \mathcal{S}(k(T-2\Delta)), \tag{30}$$

where $\mathcal{S}(k(T-2\Delta))$ is the scrap value of the savings account $k(T-2\Delta)$ that he leaves to the second-oldest self,

$$\begin{aligned} \mathcal{S}(\cdot) &= \int_{T-2\Delta}^{T-\Delta} F(t-(T-3\Delta)) \ln c^{2\Delta}(t) dt + \int_{T-\Delta}^{T} F(t-(T-3\Delta)) \ln c^{\Delta}(t) dt \\ &= \int_{T-2\Delta}^{T-\Delta} F(t-(T-3\Delta)) \ln \left[\frac{k(T-2\Delta) F(t-(T-2\Delta))}{\int_{T-2\Delta}^{T} F(t-(T-2\Delta)) dt} \right] dt \\ &+ \int_{T-\Delta}^{T} F(t-(T-3\Delta)) \ln \left[\frac{k(T-\Delta) F(t-(T-\Delta))}{\int_{T-\Delta}^{T} F(t-(T-\Delta)) dt} \right] dt, \end{aligned} \tag{31}$$

subject to the constraints

$$\frac{dk(t)}{dt} = -c(t), \tag{32}$$

$$k(T-3\Delta) \text{ given}, \tag{33}$$

$$k(T-2\Delta) \text{ free}. \tag{34}$$

The first order necessary conditions are

$$F(t-(T-3\Delta)) \frac{1}{c(t)} - \lambda(t) = 0, \tag{35}$$

$$\frac{d\lambda(t)}{dt} = 0, \tag{36}$$

$$\lambda(T-2\Delta) = \mathcal{S}'(k(T-2\Delta)). \tag{37}$$

From (36) we see that $\lambda(t)$ is constant and hence combining this result with (37) we have

$$\lambda(t) = \lambda(T-2\Delta) = \mathcal{S}'(k(T-2\Delta)), \tag{38}$$

and hence from (35) we have

$$c(t) = F(t-(T-3\Delta)) \frac{1}{\mathcal{S}'(k(T-2\Delta))}. \tag{39}$$

In order to compute $\mathcal{S}'(k(T-2\Delta))$, we first need to express the amount that the second-oldest self leaves behind, $k(T-\Delta)$ in (31), as a function of the amount he inherits $k(T-2\Delta)$. Note that (27) expresses $k(T-\Delta)$ as a function of $k(T-2\Delta)$, so we can insert (27) into (31). After doing this,

$$\begin{aligned}\mathcal{S}(\cdot) \quad = \quad & \int_{T-2\Delta}^{T-\Delta} F(t-(T-3\Delta))\ln\left[\frac{k(T-2\Delta)F(t-(T-2\Delta))}{\int_{T-2\Delta}^{T}F(t-(T-2\Delta))dt}\right]dt \\ & + \int_{T-\Delta}^{T} F(t-(T-3\Delta))\ln\left[\frac{F(t-(T-\Delta))}{\int_{T-\Delta}^{T}F(t-(T-\Delta))dt}\right]dt \\ & + \int_{T-\Delta}^{T} F(t-(T-3\Delta))\ln\left[\frac{k(T-2\Delta)\int_{T-\Delta}^{T}F(t-(T-2\Delta))dt}{\int_{T-2\Delta}^{T}F(t-(T-2\Delta))dt}\right]dt. \end{aligned} \tag{40}$$

Now, compute $\mathcal{S}'(k(T-2\Delta))$

$$\mathcal{S}'(k(T-2\Delta)) = k(T-2\Delta)^{-1}\int_{T-2\Delta}^{T} F(t-(T-3\Delta))dt. \tag{41}$$

Solving (32) with its boundary condition $k(T-3\Delta)$ we have

$$k(t) = k(T-3\Delta) - \int_{T-3\Delta}^{t} c(z)dz. \tag{42}$$

Evaluate (42) at $t = T-2\Delta$

$$k(T-2\Delta) = k(T-3\Delta) - \int_{T-3\Delta}^{T-2\Delta} c(t)dt. \tag{43}$$

Insert (39) into (43)

$$k(T-2\Delta) = k(T-3\Delta) - \int_{T-3\Delta}^{T-2\Delta} F(t-(T-3\Delta))\frac{1}{\mathcal{S}'(k(T-2\Delta))}dt, \tag{44}$$

and then insert (41) into (44)

$$k(T-2\Delta) = k(T-3\Delta) - \frac{k(T-2\Delta)}{\int_{T-2\Delta}^{T}F(t-(T-3\Delta))dt}\int_{T-3\Delta}^{T-2\Delta} F(t-(T-3\Delta))dt. \tag{45}$$

Now solve (45) for $k(T-2\Delta)$

$$k(T-2\Delta) = k(T-3\Delta)\frac{\int_{T-2\Delta}^{T}F(t-(T-3\Delta))dt}{\int_{T-3\Delta}^{T}F(t-(T-3\Delta))dt}. \tag{46}$$

Finally, combine (41) and (46)

$$\mathcal{S}'(k(T-2\Delta)) = k(T-3\Delta)^{-1}\int_{T-3\Delta}^{T} F(t-(T-3\Delta))dt, \tag{47}$$

and then insert (47) into (39)

$$c^{3\Delta}(t) = k(T-3\Delta) \times \left(\int_{T-3\Delta}^{T} F(t-(T-3\Delta))dt\right)^{-1} \times F(t-(T-3\Delta)). \tag{48}$$

9.4 Summarizing the equilibrium of the dynamic game

Problem Δ: the oldest self chooses consumption over the interval $t \in [T-\Delta, T]$,

$$c^{\Delta}(t) = k(T-\Delta) \times \left(\int_{T-\Delta}^{T} F(t-(T-\Delta))dt\right)^{-1} \times F(t-(T-\Delta)). \tag{49}$$

Problem 2Δ: the second-oldest self chooses consumption over the interval $t \in [T-2\Delta, T-\Delta]$,

$$c^{2\Delta}(t) = k(T-2\Delta) \times \left(\int_{T-2\Delta}^{T} F(t-(T-2\Delta))dt\right)^{-1} \times F(t-(T-2\Delta)). \tag{50}$$

Problem 3Δ: the third-oldest self chooses consumption over the interval $t \in [T-3\Delta, T-2\Delta]$,

$$c^{3\Delta}(t) = k(T-3\Delta) \times \left(\int_{T-3\Delta}^{T} F(t-(T-3\Delta))dt\right)^{-1} \times F(t-(T-3\Delta)). \tag{51}$$

Noting the pattern, the nth-oldest self chooses consumption over the interval $t \in [T-n\Delta, T-(n-1)\Delta]$

$$c^{n\Delta}(t) = k(T-n\Delta) \times \left(\int_{T-n\Delta}^{T} F(t-(T-n\Delta))dt\right)^{-1} \times F(t-(T-n\Delta)). \tag{52}$$

Following the pattern but working our way forward, starting with $n = T/\Delta$, then $n = T/\Delta - 1$, and then $n = T/\Delta - 2$,

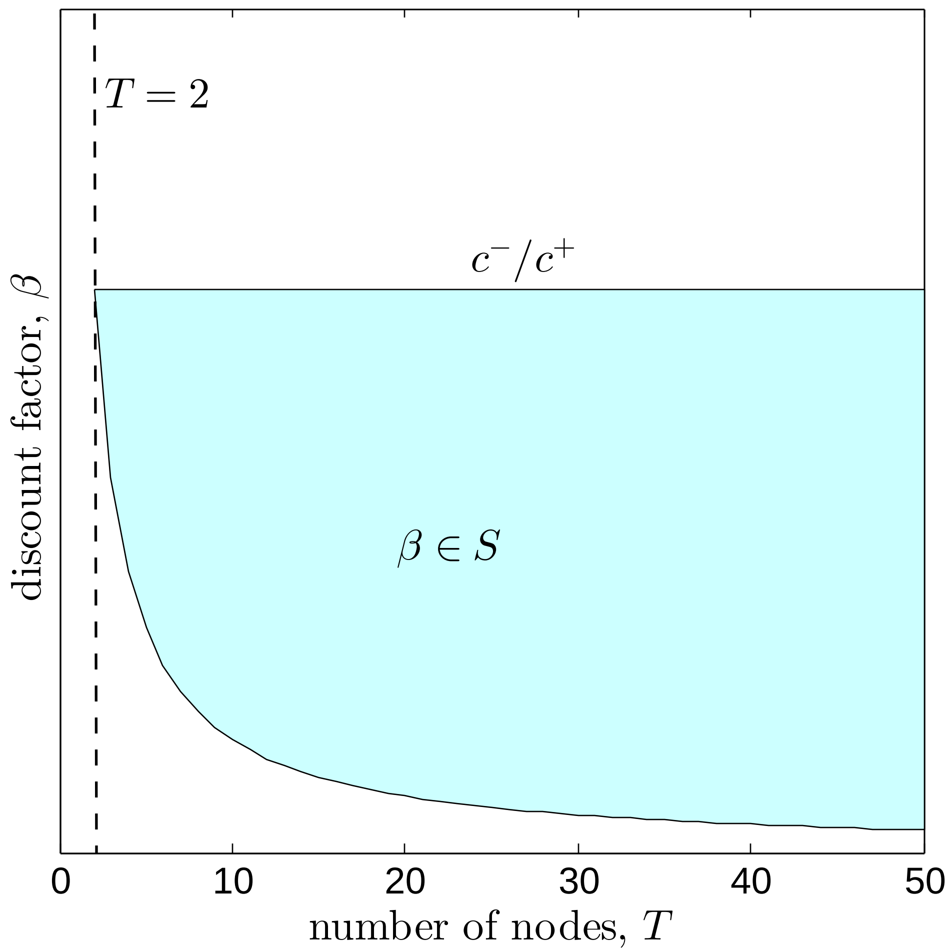
$$c(t) = k(0) \times \left(\int_{0}^{T} F(t)dt\right)^{-1} \times F(t), \text{ for } t \in [0, \Delta], \tag{53}$$

$$c(t) = k(\Delta) \times \left(\int_{\Delta}^{T} F(t-\Delta)dt\right)^{-1} \times F(t-\Delta), \text{ for } t \in [\Delta, 2\Delta], \tag{54}$$

$$c(t) = k(2\Delta) \times \left(\int_{2\Delta}^{T} F(t-2\Delta)dt\right)^{-1} \times F(t-2\Delta), \text{ for } t \in [2\Delta, 3\Delta], \tag{55}$$

and so on.

Figure 1. Parameter Space where $U(t, \mathbf{c^0}) > U(t, \mathbf{c}^*)$ for all t



Note: $U(t, \mathbf{c^0}) > U(t, \mathbf{c}^*)$ for all t, if $\beta \in S$.

Figure 2. 3 Allocations and 2 Decision Nodes ($T = 2$)

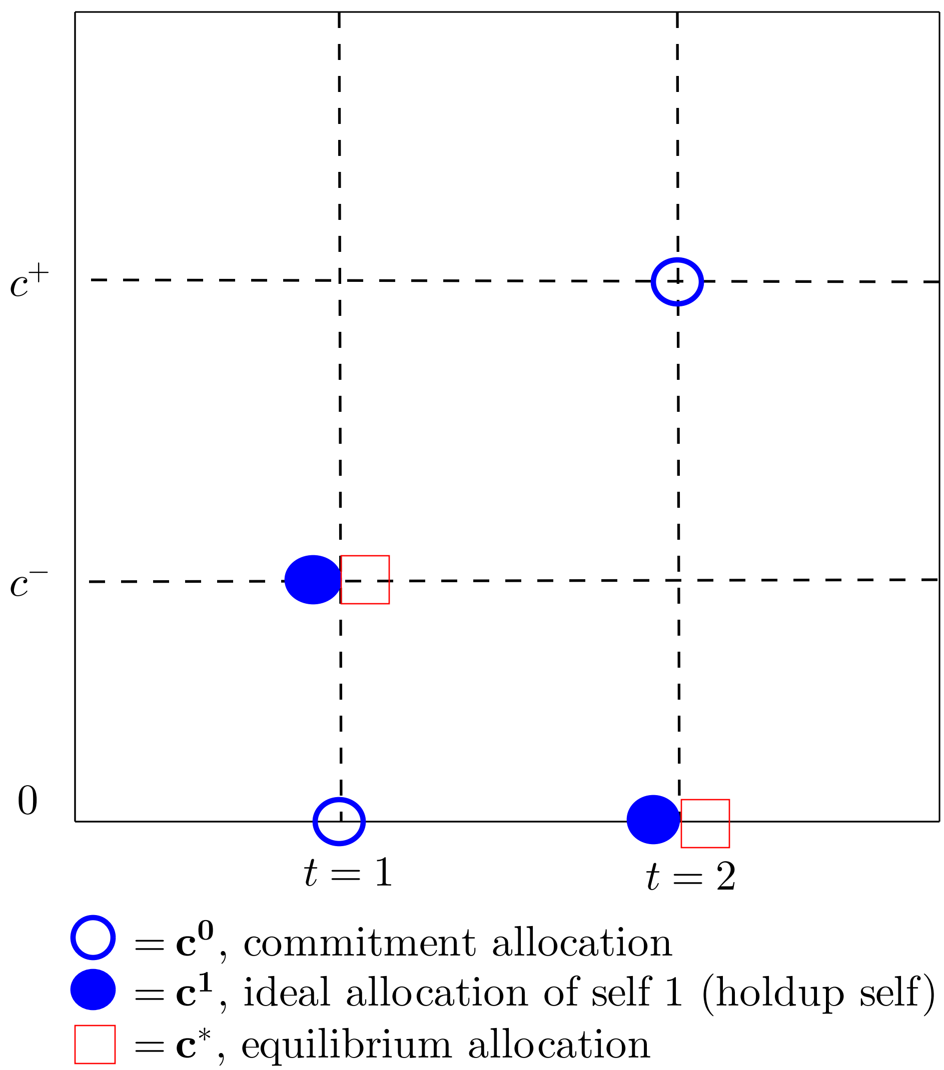
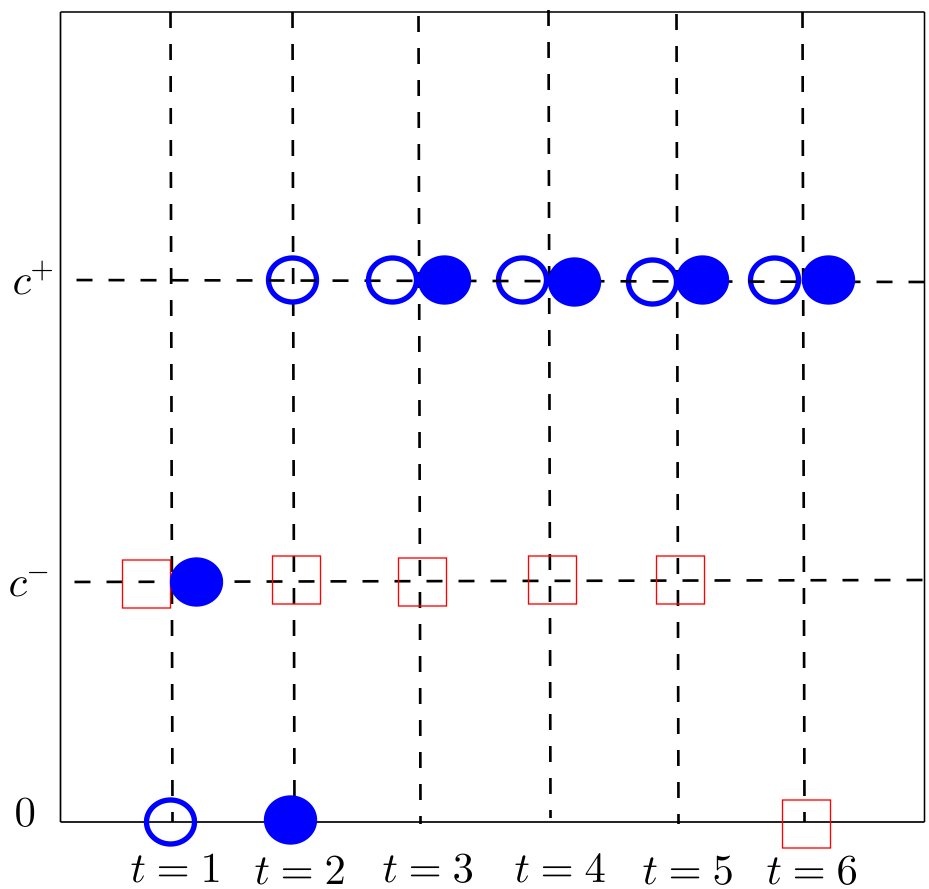
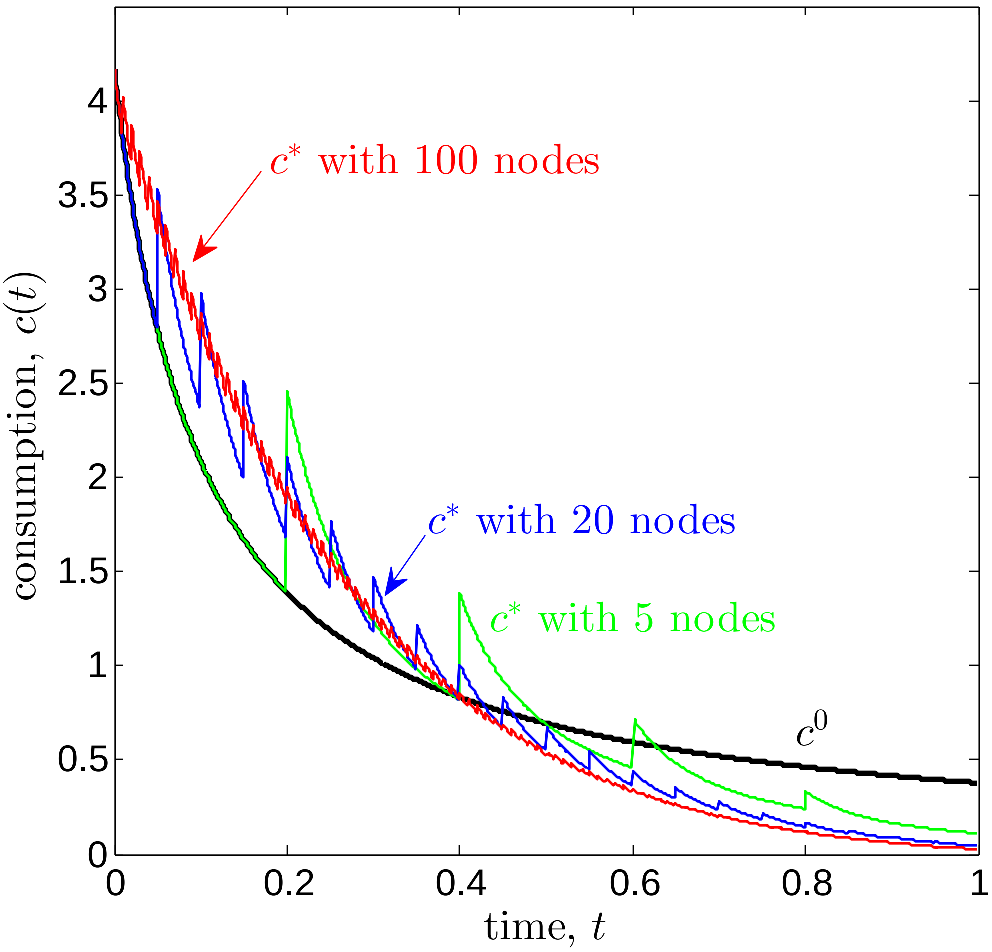


Figure 3. 3 Allocations and 6 Decision Nodes ($T = 6$)



$= \mathbf{c^0}$, commitment allocation
$= \mathbf{c^1}$, ideal allocation of self 1 (holdup self)
$= \mathbf{c}^*$, equilibrium allocation

Figure 4. Commitment and Equilibrium Consumption Allocations



Note: commitment and equilibrium allocations are c^0 and c^*.